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Exact solutions for *A–D* Temperley–Lieb models

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Abstract. We solve for the spectrum of quantum spin chains based on representations of the Temperley–Lieb algebra associated with the quantum groups $\mathcal{U}_q(X_n)$ for $X_n = A_1, B_n, C_n$ and D_n. We employ a generalization of the coordinate Bethe *ansatz* developed previously for the deformed biquadratic spin-one chain. As expected, all of these models have equivalent spectra, i.e. they differ only in the degeneracy of their eigenvalues. This is true for finite length and open boundary conditions. For periodic boundary conditions the spectra of the lower-dimensional representations are contained entirely in the higher-dimensional ones. The Bethe states are highest-weight states of the quantum group, except in certain cases where the states have energy zero.

1. Introduction

The recent interplay between the field of solvable two-dimensional lattice (or quantum spin-chain) models and quantum groups has generated a lot of interesting results. One particular way of building models which are quantum group invariant uses the Temperley–Lieb (TL) algebra [1] satisfied by the Hamitonian density U_k:

$$\begin{aligned} &U_k^2 = \sqrt{Q}U_k \qquad && U_kU_{k\pm1}U_k = U_k \\ &[U_k, U_l] = 0 \qquad && |k-l| \geqslant 2. \end{aligned} \tag{1.1}$$

The Hamiltonian is now given by the following sum over N sites:

$$H(q) = \sum_{k=1}^{N} U_k. \tag{1.2}$$

In a previous paper [2], we solved one particular example of a spin-one model, which satisfies the TL algebra. The Hamiltonian density acts on nearest neighbours as a projector on spin zero. The usual version of the coordinate Bethe *ansatz* (BA) [3, 4] does not work here and a new one had to be developed for this case [2]. It turns out that a whole host of quantum-group-invariant models are solvable by this technique, once they are recast in a common group-theoretical language.

Therefore, in section 2, we describe the representations of the TL algebra, constructed as projectors on the total spin zero of two neighbouring spins. In section 3, we introduce

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the modified coordinate BA for periodic boundary conditions and show why the usual BA, developed for the spin-1/2 XXZ-model, does not work here. Section 3 contains the algebraic details of the computation for periodic boundary conditions. We restrict ourselves to two-body sectors, because all the interesting features occur here, as is to be expected for models satisfying Yang–Baxter equations [5], as all TL models do. Of course, higher sectors have to be investigated, in order to convince oneself that the BA really does work. But since this is completely analogous to the spin-one model treated in [2] and can be applied *ipsis literis* to the present situation, once all models are described in the same language, we omit these details here. Although periodic boundary conditions simplify the application of the BA, free boundary conditions are the natural ones for quantum-group-invariant models. Hence in section 4, we spell out the differences in treating them. Section 5 is reserved for the conclusions.

2. Representations of the Temperley–Lieb algebra as spin-zero projectors

Representations of the TL algebra, commuting with quantum groups, can be constructed in the following way [6]. Suppose $\mathcal{U}_q(X_n)$ is the universal enveloping algebra of a finite-dimensional Lie algebra X_n, equipped with the coproduct $\Delta\colon \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$ [7]. If now $\pi\colon \mathcal{U}_q \to \operatorname{End} V_\Lambda$ is a finite-dimensional irreducible representation with highest weight Λ and we assume that the decomposition $V_\Lambda \otimes V_\Lambda$ is multiplicity free and includes one trivial representation on V_0, then the projector $\mathcal{P}_0$ from $V_\Lambda \otimes V_\Lambda$ onto V_0 is a representation of the TL algebra. The deformation parameter q, which plays the role of a coupling constant in the Hamiltonian, is related to Q via

$$\sqrt{Q} = \operatorname{Tr}_V(q^{-2\rho}) \tag{2.1}$$

where ρ is half the sum of the positive roots.

By construction $\mathcal{P}_0$ commutes with the quantum group $\mathcal{U}_q(X_n)$.

Since we are not going to use any group-theoretical machinery, we will just lift the relevant formulas from Batchelor and Kuniba [8] in order to display explicitly the Hamiltonians to be diagonalized.

We will consider the following specific cases: $(V_\Lambda, \mathcal{U}_q(X_n)) = (V_{2s\Lambda_1}, \mathcal{U}_q(A_1))$ for spin s, $(V_{\Lambda_1}, \mathcal{U}_q(B_n))$, $(V_{\Lambda_1}, \mathcal{U}_q(C_n))$ and $(V_{\Lambda_1}, \mathcal{U}_q(D_n))$. That is, we treat the q-deformations of the spin-s representation of sl(2) and the vector representations of so$(2n+1)$, sp$(2n)$ and so$(2n)$. V_Λ denotes the $\mathcal{U}_q(X_n)$-module with highest weight Λ. Λ_1 is a highest weight of X_n.

We introduce the following notation. Let $e_i, i = 1, \ldots, n$, be orthonormal vectors, and express the fundamental weight, $\Pi = \Lambda_1 + \ldots + \Lambda_n$, the set $\mathcal{A}$ of weights, and the coupling constant $\sqrt{Q} \equiv -2\Delta$ as

$$\begin{aligned}
&A_1\colon \mathcal{A} = \{s(e_1 - e_2), (s-1)(e_1 - e_2), \ldots, -s(e_1 - e_2)\} \\
&\Lambda_1 = (e_1 - e_2)/2 \qquad \rho = (e_1 - e_2)/2 \qquad J = \{s, s-1, \ldots, -s\} \\
&\epsilon(\mu) = (-1)^{\tilde{\mu}} \qquad \sqrt{Q} = [2s+1]
\end{aligned} \tag{2.2a}$$

$$\begin{aligned}
&B_n\ (n \geqslant 2)\colon \mathcal{A} = \{0, \pm e_1, \ldots, \pm e_n\} \\
&\Lambda_i = \begin{cases} e_1 + \ldots + e_i & (1 \leqslant i < n) \\ (e_1 + \ldots + e_n)/2 & (i = n) \end{cases} \\
&\rho = (n - 1/2)e_1 + \ldots + e_n/2 \qquad J = \{0, \pm 1, \ldots, \pm n\} \\
&\epsilon(\mu) = (-1)^{\tilde{\mu}} \qquad \sqrt{Q} = [2n-1][n+1/2]/[n-1/2]
\end{aligned} \tag{2.2b}$$

$$\begin{aligned} &C_n{:}\mathcal{A} = \{\pm e_1, \ldots, \pm e_n\} \\ &\Lambda_i = e_1 + \ldots + e_i \qquad (1 \leqslant i \leqslant n) \\ &\rho = ne_1 + \ldots + e_n \qquad J = \{\pm 1, \ldots, \pm n\} \\ &\epsilon(\mu) = \mathrm{sign}(\mu) \qquad \sqrt{Q} = [n][2n+2]/[n+1] \end{aligned} \tag{2.2c}$$

$$\begin{aligned} &D_n{:}\mathcal{A} = \{\pm e_1, \ldots, \pm e_n\} \\ &\Lambda_i = \begin{cases} e_1 + \ldots + e_i & (1 \leqslant i < n-1) \\ (e_1 + \ldots + e_{n-1} - e_n)/2 & (i = n-1) \\ (e_1 + \ldots + e_{n-1} + e_n)/2 & (i = n) \end{cases} \\ &\rho = (n-1)e_1 + \ldots + e_{n-1} \qquad J = \{0, \pm 1, \ldots, \pm n\} \\ &\sqrt{Q} = [2n-2][n]/[n-1] \qquad \epsilon(\mu) = 1. \end{aligned} \tag{2.2d}$$

For $\mu \in J$ the symbol $\tilde{\mu}$ is defined as $\tilde{\mu} = \mu + (1 \pm 1)/4$ for A_1 with $s \in \mathbb{Z} + (1 \pm 1)/4$ and $\tilde{\mu} = 0$ with the exception of $\tilde{0} = 1$ for B_n. The q-number notation is

$$[x] \equiv (q^x - q^{-x})/(q - q^{-1}).$$

For $X_n = B_n, C_n, D_n$, we extend the subscript of e_μ to $-n \leqslant \mu \leqslant n$ by setting $e_{-\mu} = -e_\mu$ (and hence $e_0 = 0$). Using the index set J above, we can write $\mathcal{A} = \{\mu(e_1 - e_2)\}$ for A_1 and $\mathcal{A} = \{e_\mu | \mu \in J)\}$ for B_n, C_n, D_n.

Denoting the unit matrix having all elements zero, except at row μ and column ν, by $E_{\mu\nu} \in \mathrm{End}\, V_\Lambda$, the projector can be written as

$$\mathcal{P}_0 = Q^{-1/2} \sum_{\mu,\nu \in J} \epsilon(\mu)\epsilon(\nu) q^{-\langle e_\mu + e_\nu, \rho\rangle} E_{\mu\nu} \otimes E_{-\mu-\nu}. \tag{2.3}$$

In the following we will refer to all of the models generically as *higher-spin* models for simplicity, even when not talking about A_1.

If we consider then a one-dimensional chain of length N with a 'spin' at each site, the spin variables range over the set of weight vectors $v_\mu | \mu \in J$ and our Hilbert space is an N-fold tensor product $V_\Lambda \otimes \ldots \otimes V_\Lambda$. For A_1, these are the q-analogues of the usual spin states.

The Hamiltonian densities acting on two neighbouring sites are then given by

$$\langle k, l | U | i, j \rangle = \epsilon(i)\epsilon(k) q^{-\langle e_i + e_k, \rho\rangle} \delta_{i+j,0}\, \delta_{k+l,0}. \tag{2.4}$$

Having now built common ground for all of the models, whose salient feature is that they are spin-zero projectors, we may now follow the steps of [2] to find their spectra.

3. The coordinate Bethe *ansatz*

All of the above Hamiltonians are U(1) invariant and we can classify their spectra according to sectors. For $A_1 (s = 1)$ the commuting operator is the total spin $S^z = \sum_{k=1}^N S_k^z$ and we set the conserved quantum number $r = N - S^z$. In general it is $r = N * \omega - S^z$ for A_1 and B_n and $r = N * (\omega - 1/2) - S^z$ for C_n and D_n. We set $\omega = \max J$.

Therefore, there exists a reference state $|\Omega\rangle$, satisfying $H|\Omega\rangle = E_0 |\Omega\rangle$, with $E_0 = 0$. We take $|\Omega\rangle$ to be $|\Omega\rangle = \prod_k^N |\omega, k\rangle$.

In every sector r there are eigenstates degenerate with $|\Omega\rangle$. They contain a set of *impurities*. We call any state obtained by *lowering* some of the $|\omega, k\rangle$s, such that the sum of any two neighbouring spins is non-zero, an impurity. Since $H(q)$ is a projector on spin

zero, all these states are annihilated by $H(q)$. In particular, they do not *move* under the action of $H(q)$, which is the reason for their name.

We will now start to diagonalize $H(q)$ in every sector. Nothing interesting happens in the sector $r < 2\omega$. The sector $r = 2\omega$ is more interesting, although still trivial, since it contains one free pseudoparticle. The main result of this paper is showing that $H(q)$ can be diagonalized in a convenient basis, constructed from products of single-pseudoparticle wavefunctions. The energy eigenvalues will be parametrized as sums of single-pseudoparticle contributions.

3.1. The sector $r = 2\omega$, containing one pseudoparticle

Starting with $r = 2\omega$, we encounter the situation where the states $|j, k\rangle$ and $|-j, k \pm 1\rangle$, $j \neq \omega$ occur in neighbouring pairs. They move under the action of $H(q)$ and mix with states containing one $|-\omega, k\rangle$. Eigenstates are superpositions of $|x_{[-\omega]}\rangle = (\ldots \omega\omega\omega \underset{x}{-\omega} \omega\omega \ldots)$ and $|x_{[j,-j]}\rangle = (\ldots \omega\omega\omega \underset{x}{+j} -j\omega\omega \ldots)$, i.e.

$$|2\omega; \ldots\rangle = \sum_x \left\{ a_\omega(x)\,|x_{[-\omega]}\rangle + \sum_j{}' b_j(x)\,|x_{[j,-j]}\rangle \right\} \tag{3.1}$$

where $\sum_j'$ means $j \in J^* = J - \{\pm\omega\}$ and the ellipses stand for parameters that the eigenvector is going to depend on. When $H(q)$ now acts on $|2\omega; \ldots\rangle$ it sees the reference configuration, except in the vicinity of x, and we obtain the eigenvalue equations

$$\begin{aligned}(E - q^{2\langle\epsilon_\omega,\rho\rangle} - q^{-2\langle\epsilon_\omega,\rho\rangle})\, a_\omega(x) \\ = a_\omega(x+1) + a_\omega(x-1) + \sum_l{}' \epsilon(\omega)\epsilon(l) q^{-\langle e_\omega + e_l, \rho\rangle}\, b_l(x-1) \\ + \sum_l{}' \epsilon(-\omega)\epsilon(l) q^{-\langle e_{-\omega} + e_l, \rho\rangle}\, b_l(x)\end{aligned} \tag{3.2a}$$

$$\begin{aligned}E b_j(x) = \epsilon(\omega)\epsilon(l) q^{-\langle e_j + e_\omega, \rho\rangle}\, a_\omega(x+1) + \epsilon(-\omega)\epsilon(l) q^{-\langle e_j - e_{-\omega}, \rho\rangle}\, a_\omega(x) \\ + \sum_l{}' \epsilon(j)\epsilon(l) q^{-\langle e_j + e_l, \rho\rangle} b_l(x) \qquad j \in J^*.\end{aligned} \tag{3.2b}$$

Eliminating the b_js, we get an equation very similar to the XXZ-model:

$$\left(E - \sum_{j\in J} q^{-2\langle e_l, \rho\rangle}\right) a_\omega(x) = a_\omega(x+1) + a_\omega(x-1). \tag{3.3}$$

We will treat periodic boundary conditions maintaining translational invariance in the following sections. They demand that $a_\omega(x+N) = a_\omega(x)$ and $b_j(x+N) = b_j(x)$. We parametrize as follows: $a_\omega(x) = a_\omega \xi^x$ and $b_l(x) = b_l \xi^x, l \in J^*$. Substituting this into (3.2) we get two eigenstates and their energies:

$$a_w = \epsilon(-\omega) q^{\langle e_\omega, \rho\rangle} + \epsilon(\omega) q^{-\langle e_\omega, \rho\rangle} \xi^{-1} \equiv \Gamma(\xi^{-1}) \tag{3.4}$$

$$\times b_l = \epsilon(l) q^{-\langle e_l, , \rho\rangle} \qquad l \in J^* \tag{3.5}$$

$$E_1 = \sum_l{}' q^{-\langle e_l, \rho\rangle} + \Gamma(\xi)\Gamma(\xi^{-1}) = \sum_{l\in J} q^{-2\langle e_l, \rho\rangle} + \xi + \xi^{-1}$$

and a highly degenerate solution with $E_2 = 0$, with the following constraint on the parameters:

$$\sum_l{}' \epsilon(l)\epsilon(\omega) q^{-2\langle e_{l+\omega}, \rho\rangle} b_l + \Gamma(\xi)\epsilon(\omega) q^{-\langle e_\omega, \rho\rangle} a_\omega = 0. \tag{3.6}$$

Here $\xi = e^{i\theta}$, θ being the momentum determined from the periodic condition to be: $\theta = 2\pi l/N$, with l an integer.

We describe this situation by saying that we have two types of pseudoparticle with energies E_1 and E_2. Whereas the pseudoparticle $|2\omega;\theta\rangle_2$ is degenerate with $|\Omega\rangle$, i.e. it propagates with energy $E_2 = 0$, the pseudoparticle $|2\omega;\theta\rangle_1$ propagates with energy

$$E_1 = -2\Delta + 2\cos\theta \qquad 2\Delta \equiv -\sum_{l\in J} q^{-2\langle e_l,\rho\rangle}. \tag{3.7}$$

As mentioned before, the energy eigenvalues are going to be parametrized as sums of single-pseudoparticle energies. Thus we write

$$E = \sum_{n=1}^{p} \epsilon_n \left({\sum_l}' q^{-2\langle e_l,\rho\rangle} + \Gamma(\xi_n)\Gamma(\xi_n^{-1}) \right) \tag{3.8}$$

where ϵ_n depends on which pseudoparticle we use: $\epsilon_n = 1$ for $E = E_1$ and $E = E_2 = 0$.

3.2. Two pseudoparticles and the XXZ Bethe ansatz

The next higher sector would be the $r = 2\omega + 1$ sector, but let us treat the $r = 4\omega$ sector first, since then we can compare it with the first non-trivial sector in the XXZ-model.

This sector contains states which each consist of two interacting pseudoparticles. We seek these eigenstates in the form

$$\begin{aligned} |4\omega;\ldots\rangle_{\epsilon_1\epsilon_2} = \sum_{x_1<x_2} \Bigg\{ & a_{\omega\omega}(x_1,x_2)|x_{1[-\omega]}, x_{2[-\omega]}\rangle + {\sum_i}' b_{\omega i}(x_1,x_2)|x_{1[-\omega]}, x_{2[i,-i]}\rangle \\ & + {\sum_j}' b_{j\omega}(x_1,x_2)|x_{1[j,-j]}, x_{2[-\omega]}\rangle + {\sum_i}' {\sum_j}' b_{ij}(x_1,x_2)|x_{1[i,-i]}, x_{2[j,-j]}\rangle \Bigg\}. \end{aligned} \tag{3.9}$$

Translational invariance now specifies $a_{\omega\omega}(x_1,x_2) = \xi^{x_1} a_{\omega\omega}(n)$ and similarly for the other wavefunctions, where $n = x_2 - x_1$. Periodic boundary conditions require that

$$\begin{aligned} & a_{\omega\omega}(n) = \xi^n a_{\omega\omega}(N-n) \\ & b_{i\omega}(n) = \xi^n b_{\omega i}(N-n) \qquad b_{ij}(n) = \xi^n b_{ji}(N-n) \end{aligned} \tag{3.10}$$

where $\xi = \xi_1\xi_2$ ($\xi_i = e^{i\theta_i}$, $i = 1, 2$) and the total momentum is $\theta_1 + \theta_2 = 2\pi l/N$, with l an integer.

According to equation (3.8), we will parametrize the energy as

$$E = \sum_{n=1}^{2} \epsilon_n \left[{\sum_l}' q^{-2\langle e_l,\rho\rangle} + \Gamma(\xi_n)\Gamma(\xi_n^{-1}) \right]. \tag{3.11}$$

Let us take the block $\epsilon_1 = \epsilon_2 = 1$ first. We try to build two-pseudoparticle eigenstates out of translationally invariant products of one-pseudoparticle excitations at x_1 and x_2 with weight functions $D_i(x_1,x_2)$, $i = 1, 2$:

$$\begin{aligned} |4\omega;\theta_1,\theta_2\rangle_{11} = \sum_{x_1<x_2} \Bigg\{ & D_1(x_1,x_2) \left[\Gamma(\xi_1^{-1})|x_{1[-\omega]}\rangle + {\sum_i}' \epsilon(i) q^{-\langle e_i,\rho\rangle} |x_{1[i,-i]}\rangle \right] \\ & \times \left[\Gamma(\xi_2^{-1})|x_{2[-\omega]}\rangle + {\sum_j}' \epsilon(j) q^{-\langle e_j,\rho\rangle} |x_{2[j,-j]}\rangle \right] \end{aligned}$$

$$+D_2(x_1,x_2)\left[\Gamma(\xi_2^{-1})|x_{1[-\omega]}\rangle+\sum_j{}'\epsilon(j)q^{-\langle e_j,\rho\rangle}|x_{1[j,-j]}\rangle\right]$$
$$\times\left[\Gamma(\xi_1^{-1})|x_{2[-\omega]}\rangle+\sum_i{}'\epsilon(i)q^{-\langle e_i,\rho\rangle}|x_{2[i,-i]}\rangle\right]\Bigg\}.$$

Comparing this with equation (3.9) and using translational invariance, implying $D_2(n)=\xi^n D_1(N-n)$, we get

$$\begin{aligned}
a_{\omega\omega}(n)&=\Gamma(\xi_1^{-1})\Gamma(\xi_2^{-1})D(n)\\
b_{\omega i}(n)&=\epsilon(i)q^{-\langle e_i,\rho\rangle}[\Gamma(\xi_1^{-1})D_1(n)+\Gamma(\xi_2^{-1})D_2(n)]\\
b_{i\omega}(n)&=\epsilon(i)q^{-\langle e_i,\rho\rangle}[\Gamma(\xi_2^{-1})D_1(n)+\Gamma(\xi_1^{-1})D_2(n)]\\
b_{ij}(n)&=\epsilon(i)\epsilon(j)q^{-\langle e_i+e_j,\rho\rangle}D(n)\qquad 3\leqslant n\leqslant N-3
\end{aligned}\tag{3.12}$$

where $D(n)=D_1(n)+D_2(n)$.

Applying $H(q)$ to the state of (3.9), we obtain a set of coupled equations for $a_{\omega\omega}(n), b_{ij}(n)$. Following [9], we split the equations into *far* equations, where excitations do not meet, and *near* equations, containing terms where they are neighbours. The far equations are

$$\begin{aligned}
&(E-2q^{-2\langle e_\omega,\rho\rangle}-2q^{2\langle e_\omega,\rho\rangle})a_{\omega\omega}(n)\\
&\quad=(1+\xi)^{-1}a_{\omega\omega}(n+1)+(1+\xi)a_{\omega\omega}(n-1)\\
&\qquad+\sum_l{}'\epsilon(l)\epsilon(\omega)q^{-\langle e_l+e_\omega,\rho\rangle}\left[\xi^{-1}b_{l\omega}(n+1)+b_{\omega l}(n-1)\right]\\
&\qquad+\sum_l{}'\epsilon(l)\epsilon(-\omega)q^{-\langle e_l-e_\omega,\rho\rangle}[b_{l\omega}(n)+b_{\omega l}(n)]\qquad 2\leqslant n\leqslant N-2
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
&(E-q^{-2\langle e_\omega,\rho\rangle}-q^{2\langle e_\omega,\rho\rangle})b_{\omega j}(n)\\
&\quad=\xi^{-1}b_{\omega j}(n+1)+\xi b_{\omega j}(n-1)+\epsilon(j)\epsilon(-\omega)q^{-\langle e_j-e_\omega,\rho\rangle}a_{\omega\omega}(n)\\
&\qquad+\epsilon(j)\epsilon(\omega)q^{-\langle e_j+e_\omega,\rho\rangle}a_{\omega\omega}(n+1)\\
&\qquad+\sum_l{}'\epsilon(l)q^{-\langle e_l,\rho\rangle}\left[\epsilon(\omega)q^{-\langle e_\omega,\rho\rangle}\xi^{-1}b_{lj}(n+1)+\epsilon(-\omega)q^{\langle e_\omega,\rho\rangle}b_{lj}(n)\right.\\
&\qquad\left.+\epsilon(j)q^{-\langle e_j,\rho\rangle}\xi^{-1}b_{\omega l}(n)\right]\qquad 2\leqslant n\leqslant N-2
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&(E-q^{-2\langle e_\omega,\rho\rangle}-q^{2\langle e_\omega,\rho\rangle})b_{j\omega}(n)\\
&\quad=b_{j\omega}(n-1)+b_{\omega j}(n+1)+\epsilon(j)\epsilon(-\omega)q^{-\langle e_j-e_\omega,\rho\rangle}a_{\omega\omega}(n)\\
&\qquad+\epsilon(j)\epsilon(\omega)q^{-\langle e_j+e_\omega,\rho\rangle}\xi a_{\omega\omega}(n-1)\\
&\qquad+\sum_l{}'\epsilon(l)q^{-\langle e_l,\rho\rangle}\left[\epsilon(\omega)q^{-\langle e_\omega,\rho\rangle}b_{jl}(n-1)+\epsilon(-\omega)q^{\langle e_\omega,\rho\rangle}b_{jl}(n)\right.\\
&\qquad\left.+\epsilon(j)q^{-\langle e_j,\rho\rangle}b_{l\omega}(n)\right]\qquad 3\leqslant n\leqslant N-3
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
Eb_{ij}(n)&=\epsilon(\omega)q^{-\langle e_\omega,\rho\rangle}\left[\epsilon(i)q^{-\langle e_i,\rho\rangle}\xi b_{\omega j}(n-1)+\epsilon(j)q^{-\langle e_j,\rho\rangle}b_{\omega i}(n+1)\right]\\
&\quad+\epsilon(-\omega)q^{\langle e_\omega,\rho\rangle}\left[\epsilon(i)q^{-\langle e_i,\rho\rangle}b_{\omega j}(n)+\epsilon(j)q^{-\langle e_j,\rho\rangle}b_{\omega i}(n)\right]\\
&\quad+\sum_l{}'\epsilon(l)q^{-\langle e_l,\rho\rangle}\left[\epsilon(i)q^{-\langle e_i,\rho\rangle}b_{lj}(n)\right.\\
&\quad\left.+\epsilon(j)q^{-\langle e_j,\rho\rangle}b_{il}(n)\right]\qquad 3\leqslant n\leqslant N-3.
\end{aligned}\tag{3.16}$$

We already know them to be satisfied, if we parametrize $D_1(n)$ and $D_2(n)$ using plane

waves:

$$D_1(n) = \xi_2^n \tag{3.17}$$

$$D_2(n) = \xi_2^N \xi_1^n. \tag{3.18}$$

The real problem arises, of course, when pseudoparticles are neighbours, and so interact, and we have no guarantee that the total energy is a sum of single-pseudoparticle energies.

Let us now have a very quick review of the XXZ-coordinate BA in order to be able to comment on the features which are not going to survive generalizations to the present models.

The equations in the sector $r_{XXZ} = 2$ are

$$\begin{aligned}(E - 2q^{\frac{1}{2}} - 2q^{-\frac{1}{2}})a(x_1, x_2) \\ = a(x_1 + 1, x_2) + a(x_1 - 1, x_2) + a(x_1, x_2 + 1) + a(x_1, x_2 - 1)\end{aligned} \tag{3.19}$$

if x_1 and x_2 are not neighbours. In the case where they are, we get

$$(E - q^{\frac{1}{2}} - q^{-\frac{1}{2}})a(x_1, x_1 + 1) = a(x_1 - 1, x_1 + 1) + a(x_1, x_1 + 2). \tag{3.20}$$

One now supposes that the parametrization (3.9) for $s = 1/2$ solves *both* of the above equations. In this case we are allowed to set $x_2 = x_1 + 1$ in equation (3.19) and subtract it from equation (3.20), yielding the following consistency condition:

$$-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})a(x_1, x_1 + 1) = a(x_1, x_1) + a(x_1 + 1, x_1 + 1). \tag{3.21}$$

This gives the *BA equation* for the XXZ-model, determining the two-body phase shift:

$$\xi_2^N = -\frac{1 + \xi + \xi_2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}{1 + \xi + \xi_1(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}. \tag{3.22}$$

This type of procedure only works for the spin-1/2 XXZ-model, for the following reason. When the two pseudoparticles come together in a configuration like $(\ldots + + + - - + + + \ldots)$ and when H is applied to the two down spins, it gives zero, since their total S_z equals -1. But whenever two excitations approach each other and becoming neighbours, and the Hamiltonian applied to them yields a non-vanishing result, then a representation like equation (3.12) cannot solve both the *far* and *near* equations. Yet this is exactly the situation arising for higher spins. As we shall see, in this case, the representation (3.12) has to modified [2], the two-body wavefunction developing a 'discontinuity' at minimum separation. We call this the *spin-zero rule*.

Now we go back to our problem of solving the near equations. They are

$$\begin{aligned}(E - q^{-2\langle e_\omega, \rho\rangle} - q^{2\langle e_\omega, \rho\rangle})a_{\omega\omega}(1) \\ = (1 + \xi)^{-1}a_{\omega\omega}(2) + {\sum_l}' \epsilon(l)q^{-\langle e_l, \rho\rangle}\big[\epsilon(\omega)q^{-\langle e_\omega, \rho\rangle}\,\xi^{-1}b_{l\omega}(2) \\ + \epsilon(-\omega)q^{\langle e_\omega, \rho\rangle}b_{\omega l}(1)\big]\end{aligned} \tag{3.23a}$$

$$\begin{aligned}(E - q^{-2\langle e_\omega, \rho\rangle})b_{\omega j}(1) \\ = \xi^{-1}b_{\omega j}(2) + \epsilon(j)\epsilon(-\omega)q^{-\langle e_j - e_\omega, \rho\rangle}a_{\omega\omega}(1) + \epsilon(j)\epsilon(\omega)q^{-\langle e_j + e_\omega, \rho\rangle}a_{\omega\omega}(2) \\ + {\sum_l}' \epsilon(l)q^{-\langle e_l, \rho\rangle}\big[\epsilon(\omega)q^{-\langle e_\omega, \rho\rangle}\xi^{-1}b_{lj}(2) + \epsilon(j)q^{-\langle e_j, \rho\rangle}b_{\omega l}(1)\big]\end{aligned} \tag{3.23b}$$

$$\begin{aligned}(E - q^{2\langle e_\omega, \rho\rangle})b_{j\omega}(2) \\ = b_{\omega j}(3) + \epsilon(j)\epsilon(-\omega)q^{-\langle e_j - e_\omega, \rho\rangle}a_{\omega\omega}(2) + \epsilon(j)\epsilon(\omega)q^{-\langle e_j + e_\omega, \rho\rangle}\xi a_{\omega\omega}(1)\end{aligned}$$

$$+\sum_l{}' \epsilon(l) q^{-\langle e_l,\rho\rangle}\left[\epsilon(-\omega) q^{\langle e_\omega,\rho\rangle} b_{jl}(2) + \epsilon(j) q^{-\langle e_j,\rho\rangle} b_{l\omega}(2)\right] \tag{3.23c}$$

$$\begin{aligned} Eb_{ij}(2) = {} & \epsilon(\omega) q^{-\langle e_\omega,\rho\rangle}\left[\epsilon(i) q^{-\langle e_i,\rho\rangle} \xi b_{\omega j}(1) + \epsilon(j) q^{-\langle e_j,\rho\rangle} b_{i\omega}(3)\right] \\ & + \epsilon(-\omega) q^{\langle e_\omega,\rho\rangle}\left[\epsilon(i) q^{-\langle e_i,\rho\rangle} b_{\omega j}(2) + \epsilon(j) q^{-\langle e_j,\rho\rangle} b_{i\omega}(2)\right] \\ & + \sum_l{}' \epsilon(l) q^{-\langle e_l,\rho\rangle}\left[\epsilon(i) q^{-\langle e_i,\rho\rangle} b_{lj}(2)\right. \\ & \left. + \epsilon(j) q^{-\langle e_j,\rho\rangle} b_{il}(2)\right] \qquad i \neq j \end{aligned} \tag{3.23d}$$

$$\begin{aligned} Eb_{ii}(2) = {} & \epsilon(i)\epsilon(\omega) q^{-\langle e_i+e_\omega,\rho\rangle}[\xi b_{\omega i}(1) + b_{i\omega}(3)] + \epsilon(i)\epsilon(-\omega) q^{-\langle e_i-e_\omega,\rho\rangle}[b_{\omega i}(2) + b_{i\omega}(2)] \\ & + \sum_l{}' \epsilon(l) q^{-\langle e_i+e_l,\rho\rangle}[b_{li}(2) + b_{il}(2)] \\ & + \sum_l{}' \epsilon(-i)\epsilon(l) q^{-\langle e_l-e_i,\rho\rangle} \mathcal{B}_i^{(l)}. \end{aligned} \tag{3.23e}$$

Here some new states are appearing. The $\mathcal{B}_i^{(l)}$ are the wavefunctions of the states of the type $(\ldots\ \underset{x}{\omega\omega i}\ l\ -l\ \omega\omega\ldots)$, $l \neq i$. Applying $H(q)$ to them we obtain the system

$$(E - q^{2\langle e_l,\rho\rangle})\mathcal{B}_i^{(l)} = \sum_{j\in J^*, j\neq -i} \epsilon(l)\epsilon(j) q^{\langle e_l+e_j,\rho\rangle} \mathcal{B}_i^{(l)} + \epsilon(l)\epsilon(-i) q^{-\langle e_l-e_i,\rho\rangle} b_{ii}(2) \tag{3.24}$$

yielding

$$\mathcal{B}_i^{(l)} = \left[\epsilon(l)\epsilon(-i) q^{\langle e_l-e_i,\rho\rangle} \Big/ \left(E - \sum_{j\in J,\, j\neq -i} q^{-2\langle e_j,\rho\rangle}\right)\right] b_{ii}(2). \tag{3.25}$$

Eliminating $\mathcal{B}_i^{(l)}$ from equation (3.23), we get

$$\begin{aligned} & \frac{E(E+2\Delta)}{E + 2\Delta + q^{\langle e_i,\rho\rangle}}\, b_{ii}(2) \\ & \quad = \epsilon(i)\epsilon(\omega) q^{-\langle e_i+e_\omega,\rho\rangle}[\xi b_{\omega i}(1) + b_{i\omega}(3)] + \epsilon(i)\epsilon(-\omega) q^{-\langle e_i-e_\omega,\rho\rangle} \\ & \qquad \times [b_{\omega i}(2) + b_{i\omega}(2)] + \sum_l{}' \epsilon(l)\epsilon(i) q^{-\langle e_i+e_l,\rho\rangle}[b_{li}(2) + b_{il}(2)]. \end{aligned} \tag{3.26}$$

In order to solve these equations, we follow [2] and now leave the value of the wavefunctions for nearest separation as arbitrary parameters:

$$\begin{aligned} a_{\omega\omega}(1) &= \Gamma(\xi)\Gamma(\xi^{-1}) D(1) + \mathcal{F}_{a_{\omega\omega}}(1) \\ b_{\omega i}(1) &= \epsilon(i) q^{-\langle e_i,\rho\rangle}[\Gamma(\xi^{-1}) D_1(1) + \Gamma(\xi^{-1}) D_2(1)] + \mathcal{F}_{b_{\omega i}}(1) \\ b_{\omega i}(2) &= \epsilon(i) q^{-\langle e_i,\rho\rangle}[\Gamma(\xi^{-1}) D_1(2) + \Gamma(\xi^{-1}) D_2(2)] + \mathcal{F}_{b_{i\omega}}(2) \\ b_{ij}(2) &= \epsilon(i)\epsilon(j) q^{-\langle e_i+e_j\rho\rangle} D(2) + \mathcal{F}_{b_{ij}}(2). \end{aligned} \tag{3.27}$$

In order for this modification to leave the far equations still satisfied, the following conditions have to hold:

$$\begin{aligned} & (1+\xi^{-1})\mathcal{F}_{a_{\omega\omega}}(1) + \sum_l{}' \epsilon(l)\epsilon(\omega) q^{-\langle e_l+e_\omega,\rho\rangle} \mathcal{F}_{b_{\omega l}}(1) + \sum_l{}' \epsilon(l)\epsilon(-\omega) q^{-\langle e_l-e_\omega,\rho\rangle} \mathcal{F}_{b_{l\omega}}(2) = 0 \\ & \xi \mathcal{F}_{b_{\omega j}}(1) + \sum_l{}' \epsilon(l)\epsilon(-\omega) q^{-\langle e_l-e_\omega,\rho\rangle} \mathcal{F}_{b_{lj}}(2) = 0 \\ & \mathcal{F}_{b_{j\omega}}(2) + \sum_l{}' \epsilon(l)\epsilon(\omega) q^{-\langle e_l+e_\omega,\rho\rangle} \mathcal{F}_{b_{jl}}(2) = 0. \end{aligned} \tag{3.28}$$

Now using equations (3.12) and equation (3.27) in equation (3.26), we get the following equation for $\mathcal{F}_{b_{ii}}(2)$:

$$\mathcal{F}_{b_{ii}}(2) = \frac{D(2)}{E+2\Delta} \qquad i \in J^*. \tag{3.29}$$

Doing the same with equation (3.23), we get

$$\mathcal{F}_{b_{ij}}(2) = 0 \qquad i \neq j. \tag{3.30}$$

These results for $\mathcal{F}_{b_{ii}}(2), \mathcal{F}_{b_{ij}}(2)$ are reasonable. In the first case the colliding excitations satisfy the *zero-spin rule* and we get a non-zero result, whereas in the second case the rule is not satisfied and we get zero. Using this in equation (3.28), we obtain for the remaining constants

$$\begin{aligned}
\mathcal{F}_{a_{\omega\omega}}(1) &= -(q^{-2\langle e_\omega,\rho\rangle} + q^{2\langle e_\omega,\rho\rangle} + 2\Delta)\ \mathcal{F}_{b_{ii}}(2) \\
\mathcal{F}_{b_{\omega j}}(1) &= -\epsilon(j)\epsilon(-\omega)q^{-\langle e_j - e_\omega,\rho\rangle}\ \mathcal{F}_{b_{ii}}(2) \\
\mathcal{F}_{b_{j\omega}}(2) &= -\epsilon(j)\epsilon(\omega)q^{-\langle e_j + e_\omega,\rho\rangle}\ \mathcal{F}_{b_{ii}}(2) \qquad j \in J^*.
\end{aligned} \tag{3.31}$$

Substituting finally the complete parametrization into the remaining near equations, we get the following Bethe *ansatz* equation:

$$\frac{D(2)}{E+2\Delta} = \frac{\xi}{1+\xi} D(1) \tag{3.32}$$

which can also be rewritten as

$$\xi_2^N = -\frac{\xi_2[(1+\xi^{-1})\xi_2 - 2\Delta - E]}{\xi_1[(1+\xi^{-1})\xi_1 - 2\Delta - E]}. \tag{3.33}$$

Using the explicit form of the energy, the set of equations determining the spectrum are

$$\xi_2^N = -\frac{1+\xi\xi_2 - 2\Delta\xi_2}{1+\xi\xi_1 - 2\Delta\xi_1} \qquad \xi^N = 1. \tag{3.34}$$

Notice that this equation is independent of n or any other representation-specific quantities. All the models considered therefore show an equivalent spectrum, when parametrized in terms of Δ.

In particular, this is the same consistency condition as one finds for the XXZ-model, showing that for $\epsilon_1 = \epsilon_2 = 1$—even for periodic boundary conditions—the spectra of all our models are equivalent to the spectrum of the XXZ-model, if expressed in terms of Δ.

We will not discuss the other two blocks ($\epsilon_1 = 1, \epsilon_2 = 0$), and $\epsilon_1 = \epsilon_2 = 0$, since the calculations are analogous to the ones presented above. For details the reader might consult [2]. In any case, the first of the two cases does not occur for the more interesting situation of free boundary conditions. It is too asymmetric to satisfy free boundary conditions. This is why for free boundary conditions the spectrum of each of our models is equivalent to the XXZ-spectrum. The block $\epsilon_1 = \epsilon_2 = 0$ has $E = 0$ and the BA equation reduces to $\xi^N = 1$, being highly degenerate. The eigenvalue $E = 0$ also occurs in the XXZ-spectrum, albeit with different degeneracy.

3.3. One pseudoparticle and impurities

Since the set-up with pseudoparticles and impurities is a little different from the case of two pseudoparticles, we will dedicate some space to it.

The eigenstates sought for will be like

$$|2\omega+1;\ldots\rangle = \sum_{x_1<x_2}\Bigg\{a_{\omega k}(x_1,x_2)\,|x_{1[-\omega]},x_{2[k]}\rangle + \sum_i{}' b_{jk}(x_1,x_2)\,|x_{1[j,-j]},x_{2[k]}\rangle$$
$$+a_{k\omega}(x_1,x_2)\,|x_{1[-k]},x_{2[-\omega]}\rangle + \sum_j{}' b_{kj}(x_1,x_2)\,|x_{1[k]},x_{2[j,-j]}\rangle\Bigg\}. \tag{3.35}$$

Translational invariance and periodic boundary conditions impose

$$a_{\omega k}(x_1,x_2)=\xi^{x_1}a_{\omega k}(n) \qquad b_{jk}(x_1,x_2)=\xi^{x_1}b_{jk}(n) \tag{3.36}$$
$$a_{\omega k}(n)=\xi^{n}a_{k\omega}(N-n) \qquad b_{jk}(n)=\xi^{n}b_{kj}(N-n) \tag{3.37}$$

where $n=x_2-x_1$, $\xi=\xi_1\xi_2$ ($\xi_i=\mathrm{e}^{\mathrm{i}\theta_i}$, $i=1,2$) and the total momentum is $\theta_1+\theta_2=2\pi l/N$, with l an integer.

Let us take the block $\epsilon_1=1$, building eigenstates out of translationally invariant products of one-pseudoparticle excitations at x_1 and an impurity at x_2 with weight functions $D_i(x_1,x_2)$, $i=1,2$ as in the previous section. This yields the parametrizations

$$\begin{aligned} a_{\omega k}(n)&=\Gamma(\xi_1^{-1})D_1(n) & a_{k\omega}(n)&=\Gamma(\xi_1^{-1})D_2(n)\\ b_{jk}(n)&=\epsilon(j)q^{-\langle e_j,\rho\rangle}D_1(n) & b_{kj}(n)&=\epsilon(j)q^{-\langle e_j,\rho\rangle}D_2(n).\end{aligned} \tag{3.38}$$

The far equations for the impurity on the right are now

$$(E-2q^{-2\langle e_\omega,\rho\rangle}-2q^{2\langle e_\omega,\rho\rangle})a_{\omega k}(x_1,x_2)$$
$$=a_{\omega k}(x_1-1,x_2)+a_{\omega k}(x_1+1,x_2)+\sum_l{}'\epsilon(l)\epsilon(\omega)q^{-\langle e_l+e_\omega,\rho\rangle}b_{lk}(x_1-1,x_2)$$
$$+\sum_l{}'\epsilon(l)\epsilon(-\omega)q^{-\langle e_l-e_\omega,\rho\rangle}b_{lk}(x_1,x_2) \qquad x_1+2\leqslant x_2\leqslant N-x_1-2 \tag{3.39}$$

$$Eb_{jk}(x_1,x_2)=\epsilon(j)\epsilon(\omega)q^{-\langle e_j+e_\omega,\rho\rangle}a_{\omega k}(x_1+1,x_2)+\epsilon(j)\epsilon(-\omega)q^{-\langle e_j-_\omega,\rho\rangle}a_{\omega k}(x_1,x_2)$$
$$+\sum_l{}'\epsilon(j)\epsilon(l)q^{-\langle e_j+e_l,\rho\rangle}b_{lk}(x_1,x_2) \qquad x_1+3\leqslant x_2\leqslant N-x_1-3 \tag{3.40}$$

(and analogous equations for the impurity on the left). Eliminating the b-functions, we get

$$\Big(E-\sum_{l\in J}q^{-2\langle e_l,\rho\rangle}\Big)a_{\omega k}(x_1,x_2)=a_{\omega k}(x_1-1,x_2)+a_{\omega k}(x_1+1,x_2) \tag{3.41a}$$

$$\Big(E-\sum_{l\in J}q^{-2\langle e_l,\rho\rangle}\Big)a_{k\omega}(x_1,x_2)$$
$$=a_{k\omega}(x_1,x_2-1)+a_{k\omega}(x_1,x_2+1) \qquad x_1+3\leqslant x_2\leqslant N-x_1-3. \tag{3.41b}$$

We know them to be satisfied, if the energy is given by equation (3.11). The near equations require of course the by now customary treatment of modifying the *ansatz* of the wavefunctions at small separations.

The near equations for the impurity at the right are

$$(E-2q^{-2\langle e_\omega,\rho\rangle})a_{\omega k}(x,x+1)$$
$$=a_{\omega k}(x-1,x+1)+\sum_l{}'\epsilon(l)\epsilon(\omega)q^{-\langle e_l+e_\omega,\rho\rangle}b_{lk}(x-1,x+1) \tag{3.42}$$

$$
\begin{aligned}
&Eb_{jk}(x,x+2)\\
&\quad=\epsilon(j)\epsilon(\omega)q^{-\langle e_j+e_\omega,\rho\rangle}a_{\omega k}(x+1,x+2)+\epsilon(j)\epsilon(-\omega)q^{-\langle e_j-e_\omega,\rho\rangle}a_{\omega k}(x,x+2)\\
&\qquad+{\sum_l}'\epsilon(j)\epsilon(l)q^{-\langle e_j+e_l,\rho\rangle}b_{lk}(x,x+2) \qquad (3.43)\\
&Eb_{jj}(x,x+2)\\
&\quad=\epsilon(j)\epsilon(\omega)q^{-\langle e_j+e_\omega,\rho\rangle}a_{\omega j}(x+1,x+2)+\epsilon(j)\epsilon(-\omega)q^{-\langle e_j-e_\omega,\rho\rangle}a_{\omega j}(x,x+2)\\
&\qquad+\epsilon(-j)\epsilon(-\omega)q^{\langle e_j+e_\omega,\rho\rangle}a_{j\omega}(x,x+1)+\epsilon(-j)\epsilon(\omega)q^{\langle e_j-e_\omega,\rho\rangle}a_{j\omega}(x,x+2)\\
&\qquad+{\sum_l}'\epsilon(j)\epsilon(l)q^{-\langle e_j+e_l,\rho\rangle}b_{lj}(x,x+2)\\
&\qquad+{\sum_l}'\epsilon(-j)\epsilon(l)q^{\langle e_j-e_l,\rho\rangle}b_{jl}(x,x+1). \qquad (3.44)
\end{aligned}
$$

They can be solved modifying the parametrization for nearest neighbours in the usual way. The result is

$$a_{\omega k}(1)=\Gamma(\xi^{-1})\xi_2+\mathcal{F}_{a_{k\omega}} \qquad b_{jk}(2)=\epsilon(j)q^{-\langle e_j,\rho\rangle}\xi_2^2+\mathcal{F}_{b_{jk}} \qquad (3.45)$$

$$a_{k\omega}(1)=\Gamma(\xi^{-1})\xi_2^N\xi_1+\mathcal{F}_{a_{k\omega}} \qquad b_{kj}(1)=\epsilon(j)q^{-\langle e_j,\rho\rangle}\xi_2^N\xi_1+\mathcal{F}_{b_{kj}} \qquad (3.46)$$

where

$$
\begin{aligned}
&\mathcal{F}_{a_{k\omega}}=-\epsilon(\omega)q^{-\langle e_\omega,\rho\rangle}\xi_2^N \qquad \mathcal{F}_{b_{kj}}=\epsilon(k)q^{-\langle e_k,\rho\rangle}\xi_2^N\delta_{k+j,0}\\
&\mathcal{F}_{a_{\omega k}}=-\epsilon(-\omega)q^{\langle e_\omega,\rho\rangle}\xi_2 \qquad \mathcal{F}_{b_{jk}}=\epsilon(-k)q^{\langle e_k,\rho\rangle}\xi\xi_2\delta_{k-j,0}
\end{aligned} \qquad (3.47)
$$

with $\xi^N=1$, $\xi_1^{N-2}\xi^2=1$, from periodic boundary conditions.

4. Free boundary conditions

It is for free boundary conditions that the Hamiltonian $H(q)$ commutes with the quantum group $\mathcal{U}_q(X_n)$. As expected, the Bethe states are the highest-weight states of $\mathcal{U}_q(X_n)$, except in the case of certain $E=0$ states. This can easily be shown, applying the quantum group spin-raising operator to these states, following the analogous steps of [2].

The extension of the BA procedure from the periodic boundary conditions to the free case again follows exactly the lines of [2]. We will therefore only state the results for the sector with two pseudoparticles.

Take the block $\epsilon_1=\epsilon_2=1$. The nearest-approach constants to be added to the now standing waves are the same as in the periodic case, namely equations (3.30) and (3.31)—only $\mathcal{F}_{b_{ii}}(2)$ is different [2]. The BA equations are now

$$\xi_a^{2N}=\prod_{b=1,b\neq a}^{r}\frac{b(\xi_a^{-1},\xi_b)}{b(\xi_a,\xi_b)} \qquad a=1,2 \qquad (4.1)$$

where

$$b(\xi_a,\xi_b)=\frac{\xi_b}{\xi_a}[\xi_b+\xi_a^{-1}-2\Delta-E_{ab}][\xi_b^{-1}+\xi_a^{-1}-2\Delta-E_{ab}] \qquad (4.2)$$

and

$$E_{ab}=2{\sum_l}'q^{-2\langle e_l,\rho\rangle}+\Gamma(\xi_a)\Gamma(\xi_a^{-1})+\Gamma(\xi_b)\Gamma(\xi_b^{-1}). \qquad (4.3)$$

The only other block is $\epsilon_1=\epsilon_2=0, E=0$. It is again highly degenerate, with $\mathcal{F}_{b_{ii}}(2), \xi_1, \xi_2$ as free parameters†.

† Actually there is one more free parameter, called α_5 in [2].

Thus all models have *spectra equivalent to the one of the XXZ-model*.

5. Conclusion

We obtained the spectra of quantum spin-chain models, arising as representations of the Temperley–Lieb algebra associated with quantum groups. The tool is a modified version of the coordinate Bethe *ansatz*, since the simpler algebraic Bethe *ansatz* is not immediately available for these models. We find that all models have equivalent spectra, i.e. they differ at most in their degeneracies for free boundary conditions, and for periodic boundary conditions the spectra of the lower-dimensional representations are entirely contained in the higher-dimensional ones.

The energy eigenvalues are given by

$$E = \sum_{n=1}^{p} \epsilon_n(-2\Delta + \Gamma(\xi_n)\Gamma(\xi_n^{-1})) \tag{5.1}$$

where $-2\Delta = \sum_l' q^{-\langle e_l,\rho\rangle}$, and the rapidities ξ_n are solutions of the BA equations.

In the sector r we may have p pseudoparticles and $N_{\omega^*-1}, N_{\omega^*-2}, \cdots, N_{-\omega^*+1}$ impurities of the type (ω^*-1), $(\omega^*-2), \cdots, (-\omega^*+1)$, respectively, such that

$$N_{\omega^*-1} + 2N_{\omega^*-2} + \cdots + (2\omega^*-1)N_{-\omega^*+1} = r - 2\omega^* p. \tag{5.2}$$

Here $\omega^* = \omega$ for A_1 and B_n and $\omega^* = \omega - 1/2$ for C_n and D_n.

For example, for periodic boundary conditions†, the total rapidity $\xi = \xi_1\xi_2\ldots\xi_p\xi_{imp}$, $\xi_{imp} = \xi_{p+1}\xi_{p+2}\ldots\xi_{r-\sigma}$ obeys $\xi^N = 1$, and the BA equations for $E \neq 0$ are

$$\xi_a^N \xi_{imp}^2 = \prod_{b=1, b\neq a}^{r-\sigma} -\frac{\xi_a}{\xi_b}\frac{(1+(\xi_a\xi_b)^{-1}\xi_a - \epsilon_{ab})}{(1+(\xi_a\xi_b)^{-1}\xi_b - \epsilon_{ab})} \tag{5.3}$$

where $\epsilon_{ab} = E_{ab} + 2\Delta$ and σ can be fixed for each allowed case from the equations (5.2).

Although we considered only the lowest non-trivial sectors in this paper, our version of the BA works in general. This can be shown, repeating step by step the reasoning of [2], where the analogous result for the spin-one case was given. We refrain therefore from burdening the reader with this material.

The Bethe eigenstates are the highest-weight states of the quantum group—except where they are states with energy $E = 0$, for which this is not always the case. Again this result can be proven very simply following [2].

It is clear from our computations that all of these results are based entirely on the Temperley–Lieb algebraic properties of our models. From them it follows that the results satisfy the Yang–Baxter [5] equations. Therefore it should not be necessary to check the validity of the BA for sectors higher than two-body ones. Also, the equivalence of the spectra should be a consequence of just Temperley–Lieb algebraic properties. As far as we know, these statements are unproven, using only Temperley–Lieb algebraic statements as input.

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† We do not list the equations for free boundary conditions, since they are identical to those of [2].

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